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Mathematics

A complete solution of the periodic Toda problem

(completely integrable systems/Abelian integrals)

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ABSTRACT **The motion of the periodic Toda lattice is explicitly determined in terms of Abelian integrals.**

1. The periodic Toda system is a system of n particles with the Hamiltonian

$$H = 2\sum_{1}^{n} p_k^2 + \sum_{k=1}^{n} \exp[-(q_k - q_{k+1})] \quad [1.1]$$

with $q_0 \equiv q_n$, and $q_1 \equiv q_{n+1}$ to make it periodic.

Following Flaschka (refs. 1 and 2), we set

$$b_k = 2p_k \quad [1.2]$$

$$a_k = \exp\left[-\frac{1}{2}(q_k - q_{k+1})\right] \quad [1.3]$$

and observe that the Hamiltonian equations of motion are equivalent to the equations

$$\frac{db_k}{dt} = 2(a_k^2 - a_{k-1}^2) \quad [1.4a]$$

and

$$\frac{da_k}{dt} = a_k(b_{k+1} - b_k). \quad [1.4b]$$

Consider now the matrices

$$A_n^{\pm} = \begin{pmatrix} b_1 a_1 & . & . & . & .\pm a_n \\ a_1 b_1 a_2 . & . & . & . & .0 \\ 0\, a_2 b_2 a_3 & . & . & . & .0 \\ \vdots & & & & \vdots \\ \pm a_n . & . & . & . & .b_n \end{pmatrix} \quad [1.5]$$

where it is understood that all elements are zero except those in the upper right and lower left corners, and on the main and the two adjacent diagonals.

It was observed by Flaschka (1) and Hénon (3) that as the matrices $A_n^{\pm}(t)$ evolve according to the Eqs. **1.4a–1.4b**, the eigenvalues $\lambda_1^+, \ldots, \lambda_n^+$ of A_n^+ as well as the eigenvalues $\lambda_1^-, \ldots, \lambda_n^-$ of A_n^- remain unchanged. The two sets of eigenvalues are not independent, and, in fact, since $a_1 a_2 \ldots a_n = 1$, it is easily seen that

$$\det(A_n^+ - \lambda I) = (-1)^n \prod_{k=1}^{n} (\lambda - \lambda_k^+) = -(-1)^n 4$$
$$+ \det(A_n^- - \lambda I) = (-1)^n\Big[\prod_{k=1}^{n} (\lambda - \lambda_k^-) - 4\Big].$$

The eigenvalues $\lambda_1^+, \ldots, \lambda_n^+$ (or any n independent functions of these eigenvalues, e.g., the traces $\sum_{k=1}^{n} \lambda_k^{+l}$, $l = 1,2,\ldots,n$] are thus constants of motion, and since their number (n) is equal to the number of degrees of freedom, the system is completely integrable. This fact alone does not guarantee the existence of an *explicit* solution of the Hamiltonian system and it is of interest to exhibit such a solution.

2. Following the development of our recent note (4), where we discussed a system of differential equations closely related to **1.4a** and **1.4b**, we introduce the matrix

$$A_{n-1}^0 = \begin{pmatrix} b_2 a_2 0 & . & . & . & 0 \\ a_2 b_3 a_3 & . & . & . & 0 \\ 0\, a_3 b_4 & . & . & . & 0 \\ \vdots & & & & a_{n-1} \\ 0 & . & . & a_{n-1} & b_n \end{pmatrix} \quad [2.1]$$

and consider its eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$.

In ref. 4 we have shown how the knowledge of $\lambda_1^+, \ldots, \lambda_n^+$ and $\mu_1, \mu_2, \ldots, \mu_{n-1}$ allows one to reconstruct A_n^+ but with a certain degree of nonuniqueness.

The result (which carries over to the present case with almost no modifications) is as follows.

Let

$$\Delta(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k^+) + 2 \quad [2.2]$$

and let

$$\delta(\mu_s) = \frac{\Delta(\mu_s) \pm \sqrt{\Delta^2(\mu_s) - 4}}{2} \quad [2.3]$$

with the understanding that the sign of $\delta(\mu_s)$ is the same as the sign of $\Delta(\mu_s)$ [one proves easily as in ref. 4 that $\Delta^2(\mu_s) - 4 \geq 0$, $s = 1,2,\ldots,n-1$, so that $\delta(\mu_s)$ is real].

Let, furthermore,

$$P(\lambda) = \prod_{j=1}^{n-1} (\lambda - \mu_s). \quad [2.4]$$

It follows that $-\delta(\mu_s)$ and $P'(\mu_s)$ are of the same sign and that

$$a_1^2 = \sum_{s=1}^{n-1} \frac{-\delta(\mu_s)}{P'(\mu_s)}. \quad [2.5]$$

[See formula **3.11** of ref. 4.]

Introducing the distribution function $\rho(\lambda)$ by the formula

$$\rho(\lambda) = \frac{1}{a_1^2} \sum_{\mu_s < \lambda} \frac{-\delta(\mu_s)}{P'(\mu_s)} \quad [2.6]$$

we obtain $a_2, \ldots, a_{n-1}, b_2, \ldots, b_n$ by introducing polynomials $\phi(k;\lambda)$ ($\phi(1;\lambda) \equiv 1$) orthogonal with respect to $\rho(\lambda)$, i.e.,

$$\int \phi(k;\lambda)\,\phi(l;\lambda)\,d\rho(\lambda) = -\frac{1}{a_1^2}\sum_{s=1}^{n-1}\frac{\phi(k;\mu_s)\,\phi(l;\mu_s)}{P'(\mu_s)}\,\delta(\mu_s) = \delta_{k,l}$$

and then using the formulas

$$b_k = \int \lambda\phi^2(k-1;\lambda)\,d\rho(\lambda), \quad k = 2,\ldots,n\text{-}1 \quad [2.7a]$$

$$a_k = \int \lambda\phi(k;\lambda)\phi(k-1;\lambda)\,d\rho(\lambda), \quad k = 2,\ldots,n\text{-}1. \quad [2.7b]$$

The remaining entries a_n and b_1 are found from the trace formulas

$$\sum_1^n b_k = \sum_1^n \lambda_k^+$$

and

$$\sum_1^n b_k^2 + 2\sum_1^n a_k^2 = \sum_1^n \lambda_k^{+2}.$$

The aforementioned ambiguity comes from the arbitrariness of sign of the radical in formula 2.3; given λ_k^+ and μ_s, there can be as many as 2^{n-1} matrices A_n^+ consistent with these data.

3. The principal result of this note and the one which leads to the complete solution of the periodic Toda problem is the formula

$$\left(\frac{1}{2}\frac{d\mu_s}{dt}P'(\mu_s)\right)^2 = \Delta^2(\mu_s) - 4, \quad s = 1,2,\ldots,n-1. \quad [3.1]$$

Let A^0_{n-2} be the left-upper $(n-2)\times(n-2)$ corner and $\tilde{A}^0_{n-2}$ the right-lower $(n-2)\times(n-2)$ corner of A^0_{n-1}. The proof of 2.1 depends on the following two identities:

$$\frac{d}{dt}\det(A^0_{n-1} - \lambda I) = 2\{a_n^2\det(A^0_{n-2} - \lambda I) - a_1^2\det(\tilde{A}^0_{n-2} - \lambda I)\} \quad [3.2]$$

and

$$\det(A^0_{n-2} - \mu_s I)\det(\tilde{A}^0_{n-2} - \mu_s I) = a_2^2,\ldots,a_{n-1}^2 \quad [3.3]$$

for $s = 1,2,\ldots,n-1$. Their proof is by induction and will not be given here. The reader is however invited to check them for $n = 3$ and $n = 4$.

We now note (by expanding the determinant according to the first row) that

$$\det(A_n^\pm - \lambda I) = (b_1 - \lambda)\det(A^0_{n-1} - \lambda I) - a_1^2\det(\tilde{A}^0_{n-2} - \lambda I) - a_n^2\det(A^0_{n-2} - \lambda I) \pm (-1)^{n-1}2a_1,\ldots,a_n \quad [3.4]$$

and hence, using the fact that $a_1,\ldots,a_n = 1$, as well as the fact that $\det(A^0_{n-1} - \mu_s I) = 0$,

$$\det(A_n^\pm - \mu_s I) = -a_1^2\det(\tilde{A}^0_{n-2} - \mu_s I) - a_n^2\det(A^0_{n-2} - \mu_s I) \pm (-1)^{n-1}2. \quad [3.5]$$

Thus,

$$\begin{aligned}\Delta^2(\mu_s) - 4 &= \det(A_n^+ - \mu_s I)\det(A_n^- - \mu_s I)\\ &= [a_1^2\det(\tilde{A}^0_{n-2} - \mu_s I) + a_n^2\det(A^0_{n-2} - \mu_s I)]^2 - 4 \quad [3.6]\\ &= [a_1^2\det(\tilde{A}^0_{n-2} - \mu_s I) - a_n^2\det(A^0_{n-2} - \mu_s I)]^2\end{aligned}$$

where we used 3.3. Combining 3.6 with 3.2 yields 3.1.

4. It is a well-known property of polynomials that

$$\sum_{s=1}^{n-1}\frac{\mu_s^r}{P'(\mu_s)} = 0, \quad r = 1,2,\ldots,n-2,$$

and

$$\sum_{s=1}^{n-1}\frac{\mu_s^{n-1}}{P'(\mu_s)} = 1.$$

We assume nondegeneracy of the intervals of instability (see ref. 4). It then follows from 3.1 that

$$\sum_{s=1}^{n-1}\mu_s^r\frac{d\mu_s/dt}{\pm\sqrt{\Delta^2(\mu_s) - 4}} = 0, \quad r = 1,\ldots,n-2 \quad [4.1a]$$

and

$$\sum_{s=1}^{n-1}\mu_s^{n-1}\frac{d\mu_s/dt}{\pm\sqrt{\Delta^2(\mu_s) - 4}} = 2 \quad [4.1b]$$

The system 4.1a, 4.1b can be solved by the classical theory of hyper-elliptic (Abelian) integrals. It follows that the μ_s as functions of time are periodic and the periods can be explicitly determined. Each μ_s is confined to an interval either between certain pairs of consecutive λ_i^+ or certain pairs of consecutive λ_i^- (for example, if n is odd and all λ_i^+ and λ_i^- nondegenerate, one has $\lambda_1^+ \leq \mu_1 \leq \lambda_2^+$, $\lambda_2^- \leq \mu_2 \leq \lambda_3^-$, etc.), and $d\mu_s/dt$ can change sign only at the endpoints of these intervals.

We also observe that

$$\frac{\delta(\mu_s)}{P'(\mu_s)} = \frac{1}{2}\frac{\Delta(\mu_s)}{P'(\mu_s)} \pm \frac{1}{4}\frac{d\mu_s}{dt}.$$

At $t = 0$ the signs of $d\mu_s/dt$ must so be chosen as to yield an initial $\rho(\lambda)$ which is consistent with the initial data $a_k(0)$, $b_k(0)$. Once chosen, these signs persist for all times to yield an analytic evolution of $\rho(\lambda)$. If some of the λ_i^+ and (or) λ_i^- are degenerate, the corresponding μ's are constant and the number of equations in the system 4.1 is correspondingly decreased. Such degeneracies occur only if the entries of $A^\pm$ satisfy certain algebraic relations (which because of isospectrality of the Hamiltonian flow persist for all times).

Details and further discussion are reserved for a future publication.

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